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Measures of entanglement based on decoherence

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Abstract
New measures of entanglement—the time of disentanglement and the rate of decoherence—of states of a compound system are proposed. They are based on the dynamical properties of the system induced by the measurement-like interaction with an environment. In particular, it is shown that in the case of two qubits the time of disentanglement may serve as a practical measure of entanglement of mixed states.

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1. Introduction

Entanglement of quantum states is the most non-classical feature of quantum systems. It shows up when the system consists of two or more subsystems and the total state cannot be written as a product state. Mathematically there is nothing very special about this but the physical consequences are extraordinary. Indeed, an entangled state cannot be thought of as a state of a composite system in any classical sense. Entanglement has such remarkable consequences that Schrödinger was led to say that it was 'not one but rather the characteristic trait of quantum mechanics' [1]. Entangled states play a central role in quantum communication [2], cryptography [3] and quantum computing [4], so it is important to know what amount of entanglement a given quantum state contains. For any bipartite system in a pure state ψ it was argued in [5] that the *entropy of entanglement*

$$E(\psi) = -\mathrm{tr}\hat{\rho}\log_2\hat{\rho}$$

where $\hat{\rho}$ is a partial trace of $|\psi\rangle\langle\psi|$ over either one of the two subsystems, is a reasonable measure of entanglement, and it was shown in [6] that $E(\psi)$ is essentially a unique measure of entanglement for pure states. For mixed state ρ, it seems that the basic measure is the *entanglement of formation* [7]

$$E(\rho) = \min \sum_i p_i E(\psi_i)$$

where the minimum is taken over all possible decompositions

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

Since $E(\rho)$ involves a minimalization procedure, in general it cannot be easily computed analytically or even numerically. Fortunately, in the case of two qubits, an explicit formula for $E(\rho)$ exists [8,9], and $E(\rho)$ is a function of another useful quantity called *concurrence*, which also can be taken as a measure of entanglement. Another class of measures was introduced in [10, 11]. They are defined as follows: let $\mathcal{D}$ be the set of all separable states, then

$$E(\rho) = \min_{\sigma\in\mathcal{D}} D(\rho\|\sigma)$$

where D is any measure of distance between two density matrices. For appropriate distance D, the measure $E(\rho)$ can be shown to satisfy the following natural conditions: (i) $E(\rho) = 0$ iff ρ is separable; (ii) local unitary operations leave $E(\rho)$ invariant and (iii) $E(\rho)$ cannot increase under transformations involving local general measurements and classical communication [11].

In this paper, we propose measures of entanglement based on dynamical properties of a compound system, induced by measurement-like interaction with an environment. The evolution of density matrices of a compound system is given, in the Markovian approximation, by an ergodic dynamical semi-group $\{T_t\}$, i.e. a trace preserving semi-group of operators completely positive and contractive in the trace norm having the completely mixed state $\frac{1}{n}\mathbb{I}$ invariant. Such a type of evolution is a particular realization of so-called strictly contractive quantum channels [12]. The interaction between the system and its environment described by $\{T_t\}$ makes all states unstable, and for a bipartite system the class of the most unstable states coincides with the class of maximally entangled states [13]. Moreover, the degree of stability may serve as another measure of entanglement of pure states. In this paper we discuss the evolution of an arbitrary two-qubit density matrix ρ. We shall show that if ρ is separable, then for any $t \geqslant 0$, $\rho_t = T_t(\rho)$ is also separable, but when ρ is entangled (i.e. partial transposition ρ^{T_A} is not positive definite [14]), there is a definite *time of disentanglement* $t_{\mathrm{d}}(\rho)$ after which ρ_t becomes separable. We provide a formula for $t_{\mathrm{d}}(\rho)$ in the case of an arbitrary two-qubit density matrix and compute this quantity for several examples. Comparison of $t_{\mathrm{d}}(\rho)$ with measures of entanglement, and its general properties listed in theorem 1, suggest that $t_{\mathrm{d}}(\rho)$ may be considered as a kind of new measure of entanglement, with intuitive physical origin and a simple algorithm for analytic and numerical calculation. Moreover, it is shown that for a class of so-called T-states $t_{\mathrm{d}}(\rho)$ is an increasing function of the concurrence and so it is equivalent to the entanglement of formation. It is worth noting that the time of disentanglement may be also defined in the case of n-dimensional systems coupled together. In the paper we also discuss the stability properties of three- and four-qubit states. It turns out that it is useful to introduce the rate of stability of pure states expressed in terms of the pair $(\lambda(P), \lambda_{\min}(P))$ consisting of the rate of decoherence $\lambda(P)$ and minimal rate of decoherence $\lambda_{\min}(P)$. It is shown that in both three- and four-qubit cases the most entangled states are the most unstable.

2. Two-qubit states

Suppose A, B are spin-$\frac{1}{2}$ systems, i.e. they are represented by a 2×2 matrix algebra. The algebra of the joint system AB is equal to $M_{4\times4}$. We assume that the system AB is open and interacting with its environment, so the reduced dynamics of the system AB is given, in the Markovian approximation, by the master equation

$$\dot{\rho} = L\rho = -\mathrm{i}[H, \rho] + L_{\mathrm{D}}\rho \tag{1}$$

where $H = H^* \in M_{4\times4}$, and L_{D} denotes the dissipative part of the generator L of a semi-group T_t. L_{D} depends explicitly on the type of interaction between the system AB and the environment. In our model we assume that the environmental monitoring is of a measurement-like type with respect to a family of non-commuting projectors, i.e.

$$L_{\mathrm{D}}\rho = \kappa\left(\int \mathrm{d}\mu\,(x) P_x \rho P_x - \rho\right) \tag{2}$$

where $\{P_x\}$ is a continuous family of projectors such that $\int \mathrm{d}\mu\,(x) P_x = \mathbb{I}$. It is worth noting that formula (2) is a straightforward generalization of the expression $\sum_i P_i \rho P_i - \rho$ to the continuous case. In our model we take two families of projectors of the following type: $\{P_A \otimes \mathbb{I}_B\}$ and $\{\mathbb{I}_A \otimes P_B\}$, where P_A and P_B are one-dimensional projectors associated with states of the system A and B, respectively. Therefore

$$L_{\mathrm{D}}\rho = \kappa\left(\int_{S^2} \mathrm{d}\mu\,(\boldsymbol{n})\,(P_A(\boldsymbol{n}) \otimes \mathbb{I}_B\,)\,\rho\,(P_A(\boldsymbol{n}) \otimes \mathbb{I}_B\,) + \int_{S^2} \mathrm{d}\mu\,(\boldsymbol{n})\,(I_A \otimes P_B(\boldsymbol{n}))\,\rho\,(\mathbb{I}_A \otimes P_B(\boldsymbol{n})) - 2\rho\right) \tag{3}$$

where $P_{A\,(B)}(\boldsymbol{n})$ is a one-dimensional projector in $\mathbb{C}^2$ corresponding to the point $\boldsymbol{n} \in S^2 \subset \mathbb{R}^3$, and the measure $\mathrm{d}\mu\,(\boldsymbol{n})$ on S^2 is normalized in such a way that

$$\int_{S^2} \mathrm{d}\mu\,(\boldsymbol{n}) P_{A\,(B)}(\boldsymbol{n}) = \mathbb{I}_{A\,(B)}.$$

If we put $\kappa = 3/2$, then we obtain [13]

$$L_{\mathrm{D}}\rho = \frac{I_A}{2} \otimes \mathrm{tr}_A\rho + \mathrm{tr}_B\rho \otimes \frac{\mathbb{I}_B}{2} - 2\rho = \mathrm{Tr}_A\rho + \mathrm{Tr}_B\rho - 2\rho \tag{4}$$

where $\mathrm{tr}_A\,(\mathrm{tr}_B)\ :\ M_{4\times4} \to M_{2\times2}$ denotes the partial trace with respect to system $A\,(B)$ respectively and $\mathrm{Tr}_A\ (\mathrm{Tr}_B)$ is the conditional expectation from $M_{4\times4}$ onto a subalgebra of $M_{4\times4}$ isomorphic to $\mathbb{I}_A \otimes M_{2\times2}\ (M_{2\times2} \otimes \mathbb{I}_B)$.

In our previous paper [13] we studied the stability of states with respect to such a kind of interaction. We showed that the rate of decoherence of a pure state P, given by

$$\lambda(P) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} S_{\mathrm{lin}}(T_t P)\big|_{t=0} \tag{5}$$

where $S_{\mathrm{lin}}(\rho) = \mathrm{tr}\,(\rho - \rho^2)$, determines the set of maximally entangled states, as the most unstable states for which $\lambda(P) = \lambda_{\max}$. $\lambda(P)$ may also be taken as another measure of entanglement for pure states and it provides some information about entanglement of mixed states.

In this paper, we study the evolution of states of a compound system AB, given by the semi-group T_t generated by L with $H = 0$. Then

$$T_t(\rho) = \mathrm{e}^{-2t}\rho + \mathrm{e}^{-t}(1 - \mathrm{e}^{-t})\left(\frac{\mathbb{I}_A}{2} \otimes \mathrm{tr}_A\rho + \mathrm{tr}_B\rho \otimes \frac{\mathbb{I}_B}{2}\right) + (1 - \mathrm{e}^{-t})^2\,\frac{\mathbb{I}_{AB}}{4}. \tag{6}$$

This evolution has the following important property: if ρ is separable, then $T_t(\rho)$ is also separable for all $t \geqslant 0$. To show this, suppose that

$$\rho = \sum_i \alpha_i P_i \otimes Q_i \qquad \alpha_i \geqslant 0 \quad \text{and} \quad \sum_i \alpha_i = 1.$$

Then

$$T_t(\rho) = \mathrm{e}^{-2t}\rho + 2\mathrm{e}^{-t}(1 - \mathrm{e}^{-t})\left(\frac{\mathbb{I}_A}{4} \otimes \sum_i \alpha_i Q_i + \sum_i \alpha_i P_i \otimes \frac{I_B}{4}\right) + (1 - \mathrm{e}^{-t})^2\,\frac{\mathbb{I}_{AB}}{4}$$

and so $T_t(\rho)$ is a convex combination of separable density matrices, and hence is separable. This property follows also from the fact that our dynamics is local, and so it cannot entangle two particles.

Let now ρ be an arbitrary density matrix. We divide the set $[0, \infty)$ into two disjoint subsets A_ρ and B_ρ such that $A_\rho \cup B_\rho = [0, \infty)$:

$$A_\rho = \{t \in [0, \infty) \ : \ T_t(\rho) \text{ is entangled}\}$$
$$B_\rho = \{t \in [0, \infty) \ : \ T_t(\rho) \text{ is separable}\}.$$

Clearly, by the above property, for every $t_1 \in A_\rho$ and every $t_2 \in B_\rho$ we have $t_1 < t_2$. Moreover, the set B_ρ is always non-empty since any ρ evolves to a completely mixed state $\frac{1}{4}\mathbb{I}_{AB}$ and there is a neighbourhood of this state which contains only separable states.

Definition. *The time of disentanglement* $t_\mathrm{d}(\rho)$ *of a density matrix* ρ *is given by*

$$t_\mathrm{d}(\rho) = \inf \ B_\rho.$$

Since the set of separable states is compact, B_ρ is closed and $t_\mathrm{d}(\rho) \in B_\rho$. Therefore $t_\mathrm{d}(\rho)$ may be equivalently defined as the smallest time for which $T_t(\rho)$ is separable. The next theorem lists some properties of the time of disentanglement.

Theorem 1.

(1) $t_\mathrm{d}(\rho) \geqslant 0$ *and* $t_\mathrm{d}(\rho) = 0$ *if and only if* ρ *is separable.*
(2) Local unitary transformations leave t_d *invariant, i.e. for any unitary matrices* $U_1, \ U_2 \in M_{2\times 2}$

$$t_\mathrm{d}(U_1 \otimes U_2 \ \rho \ U_1^* \otimes U_2^*) = t_\mathrm{d}(\rho).$$

(3) $t_\mathrm{d}(\tilde{\rho}) \leqslant t_\mathrm{d}(\rho)$ *for* $\tilde{\rho}$ *being a convex combination of matrices of the form*

$$U_i \otimes V_i \ \rho \ U_i^* \otimes V_i^*$$

where $U_i, \ V_i$ *are unitary* 2×2 *matrices.*
(4) Suppose $A_i, \ B_j \in M_{2\times 2}$ *are such that*

$$\sum_i A_i^* A_i = \sum_i A_i A_i^* = \mathbb{I}_A$$
$$\sum_j B_j^* B_j = \sum_j B_j B_j^* = \mathbb{I}_B.$$

Then

$$t_\mathrm{d}\left(\sum_{ij} A_i \otimes B_j \ \rho \ A_i^* \otimes B_j^* \right) \leqslant t_\mathrm{d}(\rho).$$

In other words, t_d *does not increase under local general measurements.*

Proof.

(1) This follows directly from the definition.
(2) Since

$$T_t(U_1 \otimes U_2 \ \rho \ U_1^* \otimes U_2^*) = U_1 \otimes U_2 \ T_t(\rho) \ U_1^* \otimes U_2^*$$

$T_t(U_1 \otimes U_2 \ \rho \ U_1^* \otimes U_2^*)$ is separable if and only if $T_t(\rho)$ is separable.

(3) Suppose that

$$\tilde{\rho} = \sum_i \alpha_i U_i \otimes V_i \; \rho \; U_i^* \otimes V_i^*$$

where $\sum_i \alpha_i = 1$, or more generally,

$$\tilde{\rho} = \int \mathrm{d}\mu \,(U \otimes V) U \otimes V \; \rho \; U^* \otimes V^*$$

where $\mathrm{d}\mu$ is a probability measure on $U(2) \otimes U(2)$. Then

$$T_t(\tilde{\rho}) = \int \mathrm{d}\mu \,(U \otimes V) U \otimes V \; T_t(\rho) \; U^* \otimes V^*$$

and so $T_t(\tilde{\rho})$ is separable if $T_t(\rho)$ is separable.

(4) Let

$$\tilde{\rho} = \sum_{ij} A_i \otimes B_j \; \rho \; A_i^* \otimes B_j^*.$$

Suppose $T_t(\rho)$ is separable. Then [16, 17]

$$\mathbb{I}_A \otimes \mathrm{tr}_A T_t(\rho) - T_t(\rho) \geqslant 0. \tag{7}$$

Because condition (7) is also sufficient for a density matrix to be separable and

$$\mathbb{I}_A \otimes \mathrm{tr}_A T_t(\tilde{\rho}) - T_t(\tilde{\rho}) = \sum_{ij} A_i \otimes B_j \, [\mathbb{I}_A \otimes \mathrm{tr}_A T_t(\rho) - T_t(\rho)] \, A_i^* \otimes B_j^* \geqslant 0$$

$T_t(\tilde{\rho})$ is separable too. Hence $t_{\mathrm{d}}(\tilde{\rho}) \leqslant t_{\mathrm{d}}(\rho)$.

□

Using (7) we are now in position to derive a formula for the time t_{d}. We look for the smallest value of time t such that for any $\boldsymbol{v} \in \mathbb{C}^4$, $\|\boldsymbol{v}\| = 1$

$$\langle \boldsymbol{v}, \, [2\mathrm{Tr}_A T_t(\rho) - T_t(\rho)] \, \boldsymbol{v} \rangle \geqslant 0.$$

If we substitute $x = \mathrm{e}^t$ the above inequality may be written as

$$\tfrac{1}{4}x^2 + x\langle \boldsymbol{v}, \, (\mathrm{Tr}_A\rho - \mathrm{Tr}_B\rho)\boldsymbol{v} \rangle + \langle \boldsymbol{v}, \, (\rho - \mathrm{Tr}_A\rho - \mathrm{Tr}_B\rho + \mathrm{Tr}_{AB}\rho)\boldsymbol{v} \rangle \geqslant 0. \tag{8}$$

Let us denote

$$\Delta(\boldsymbol{v}) = \langle \boldsymbol{v}, \, (\mathrm{Tr}_A\rho - \mathrm{Tr}_B\rho)\boldsymbol{v} \rangle^2 + \langle \boldsymbol{v}, \, (\rho - \mathrm{Tr}_A\rho - \mathrm{Tr}_B\rho + \mathrm{Tr}_{AB}\rho)\boldsymbol{v} \rangle.$$

Because $t_{\mathrm{d}}(\rho)$ is equal to the logarithm of the greater root of the lhs of inequality (8),

$$t_{\mathrm{d}}(\rho) = \max\left(0, \; \ln\left[2 \sup_{\|v\|=1} \left(\langle \boldsymbol{v}, \, (\mathrm{Tr}_B\rho - \mathrm{Tr}_A\rho)\boldsymbol{v} \rangle + \sqrt{\Delta(\boldsymbol{v})}\right)\right]\right) \tag{9}$$

where the supremum is taken over all normalized vectors $\boldsymbol{v}$ such that $\Delta(\boldsymbol{v}) > 0$. In the particular case of states ρ for which $\mathrm{tr}_A\rho = \mathrm{tr}_B\rho = \frac{1}{2}\mathbb{I}$ (so-called T-states [15]), formula (9) simplifies to

$$t_{\mathrm{d}}(\rho) = \max\,(0, \; \tfrac{1}{2}\ln(4p_{\max}(\rho) - 1)) \tag{10}$$

where $p_{\max}(\rho)$ is the maximal eigenvalue of ρ. On the other hand, the class of T-states coincides with the class of density matrices which are real when expressed in the 'magic basis' [8]

$$f_1 = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle) \qquad f_2 = \frac{\mathrm{i}}{\sqrt{2}}(|11\rangle - |00\rangle)$$

$$f_3 = \frac{\mathrm{i}}{\sqrt{2}}(|10\rangle + |01\rangle) \qquad f_4 = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

i.e. satisfying

$$\rho = \rho^{\dagger} = (\sigma_y \otimes \sigma_y)\, \overline{\rho}\, (\sigma_y \otimes \sigma_y)$$

where $\overline{\rho}$ denotes complex conjugation of the matrix ρ. Since the concurrence $C(\rho)$ is given by [8]

$$C(\rho) = \max\left(0, 2p_{\max}(\hat{\rho}) - \mathrm{tr}\,\hat{\rho}\right)$$

where

$$\hat{\rho} = (\rho^{1/2}\, \rho^{\dagger}\, \rho^{1/2})^{1/2}$$

for the class of T-states we obtain

$$C(\rho) = \max\,(0, 2p_{\max}(\rho) - 1)\,.$$

Thus the following relation holds.

Proposition 1. *For a class of T-states, i.e. density matrices ρ satisfying* $\mathrm{tr}_A\rho = \mathrm{tr}_B\rho = \frac{1}{2}\mathbb{I}$, *the time of disentanglement $t_{\mathrm{d}}(\rho)$ is given by the following function of concurrence $C(\rho)$:*

$$t_{\mathrm{d}}(\rho) = \tfrac{1}{2}\ln(2C(\rho) + 1). \tag{11}$$

Although it is usually much simpler to calculate $t_{\mathrm{d}}(\rho)$ than the infimum of some distance $D(\rho\|\sigma)$ over the space of all separable density matrices σ, formula (9) is not very practical. In a particular case with a fixed ρ the following algorithm proved to be useful:

- calculate $T_t(\rho)$;
- take the partial transposition $T_t(\rho)^{T_A}$ with respect to subsystem A;
- solve the equation $\det T_t(\rho)^{T_A} = 0$ (after substitution $x = \mathrm{e}^{-2t}$ this equation is converted to an algebraic one of the fourth degree) and
- if there are roots $x_i \in (0, 1]$ pick up the smallest one, say x_1, and put $t_{\mathrm{d}}(\rho) = -\frac{1}{2}\ln x_1$.

  Otherwise take $t_{\mathrm{d}}(\rho) = 0$.

Let us now calculate the time of disentanglement for some families of density matrices.

Example 1. Pure states. Let

$$\psi = \begin{pmatrix} R_1 \\ R_2\mathrm{e}^{\mathrm{i}\theta_2} \\ R_3\mathrm{e}^{\mathrm{i}\theta_3} \\ R_4\mathrm{e}^{\mathrm{i}\theta_4} \end{pmatrix} \qquad R_1, R_2, R_3, R_4 \geqslant 0 \quad R_1^2 + R_2^2 + R_3^2 + R_4^2 = 1 \quad \theta_2, \theta_3, \theta_4 \in [0, 2\pi]$$

be an arbitrary pure state of a two-qubit system. By direct calculation we obtain that

$$t_{\mathrm{d}}(|\psi\rangle\langle\psi|) = \tfrac{1}{2}\ln\left[\,1 + 4\sqrt{R_1^2R_4^2 + R_2^2R_3^2 - 2R_1R_2R_3R_4\cos(\theta_2 + \theta_3 - \theta_4)}\,\right]. \tag{12}$$

In particular, the maximal value of $t_{\mathrm{d}}(|\psi\rangle\langle\psi|) = \frac{1}{2}\ln 3$ is attained for a three-parameter family of maximally entangled states. Let us now compare t_{d} with another measure of entanglement for pure states. It turns out that t_{d} and entropy $E_A = -\mathrm{tr}\,(\mathrm{tr}_A|\psi\rangle\langle\psi|)\ \log(\mathrm{tr}_A|\psi\rangle\langle\psi|)$ as well as the rate of decoherence $\lambda(|\psi\rangle\langle\psi|)$ [13] depend only on one parameter

$$\alpha = R_1^2R_4^2 + R_2^2R_3^2 - 2R_1R_2R_3R_4\cos(\theta_2 + \theta_3 - \theta_4)$$

and so

$$t_{\mathrm{d}} = \tfrac{1}{2}\ln[1 + 4\sqrt{\alpha}]$$

$$E_A = -\frac{1}{2}\left[\left(1 - \sqrt{1 - 4\alpha}\right)\ln\frac{1 - \sqrt{1 - 4\alpha}}{2} + \left(1 + \sqrt{1 - 4\alpha}\right)\ln\frac{1 + \sqrt{1 - 4\alpha}}{2}\right]$$

$$\lambda = 1 + 2\alpha.$$

Remark. By direct calculation we can check that the relation between t_d and the concurrence given by formula (11) is also valid if ρ is an arbitrary pure state.
Below we plot functions t_d, E_A and $\lambda - 1$. Clearly, all of them give the same order of entanglement among pure states.

Example 2. Bell-diagonal states. Let

$$\rho_B = p_1|\Phi^+\rangle\langle\Phi^+| + p_2|\Phi^-\rangle\langle\Phi^-| + p_3|\Psi^+\rangle\langle\Psi^+| + p_4|\Psi^-\rangle\langle\Psi^-|$$

where

$$\Phi^{\pm} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \qquad \Psi^{\pm} = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle).$$

It is known that if all $p_i \in [0, 1/2]$, ρ_B is separable, while for $p_1 > 1/2$, ρ_B is entangled (similarly for p_2, p_3, p_4) [15]. Since ρ_B belong to the class of T-states,

$$t_d(\rho_B) = \tfrac{1}{2}\ln(4p_1 - 1) \qquad p_1 > 1/2. \tag{13}$$

Let us compare (13) with the measure of entangled $E(\rho_B)$ computed in [18]

$$E(\rho_B) = p_1 \ln p_1 + (1 - p_1)\ln(1 - p_1) + \ln 2. \tag{14}$$

t_d as well as E is an increasing function of p_1 and they attain their maximal value for $p_1 = 1$ (see figure 2).

Example 3. Maximally entangled mixed states.
Recently, a class of mixed two-qubit states was discovered, which are conjectured to be maximally entangled for a given degree of inpurity measured by $\mathrm{tr}\rho^2$ [19]. They form the following family of states:

$$\rho_M = \begin{pmatrix} g(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1-2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & g(\gamma) \end{pmatrix} \qquad g(\gamma) = \begin{cases} 1/3 & \gamma \in [0, 2/3] \\ \gamma/2 & \gamma \in [2/3, 1]. \end{cases}$$

Also in this case we are able to compute the time of disentanglement, which is given by

$$t_d(\rho_M) = \begin{cases} \frac{1}{2}\ln\left(\frac{5}{9} + \frac{2}{9}\sqrt{4 + 81\gamma^2}\right) & \gamma \in [0, 2/3] \\ \frac{1}{2}\ln\left(1 - 2\gamma + 2\gamma^2 + 2\gamma\sqrt{2 - 2\gamma + \gamma^2}\right) & \gamma \in [2/3, 1]. \end{cases} \tag{15}$$

A comparison with Werner states [20]

$$\rho_W = (1 - p)\frac{\mathbb{I}_4}{4} + p|\Psi\rangle\langle\Psi|$$

where Ψ is a maximally entangled pure state, shows that for a fixed $\mathrm{tr}\rho^2$, $t_d(\rho_M) > t_d(\rho_W) = \frac{1}{2}\ln 3p$ (see figure 3). Moreover, computer simulations show that indeed $t_d(\rho_M)$ is a maximal possible value of t_d for fixed $\mathrm{tr}\rho^2$. Let us point out that maximally entangled mixed states with $\gamma \in (0, 1)$ provide an example of states for which $t_d(\rho) < \frac{1}{2}\ln(2C(\rho) + 1)$.

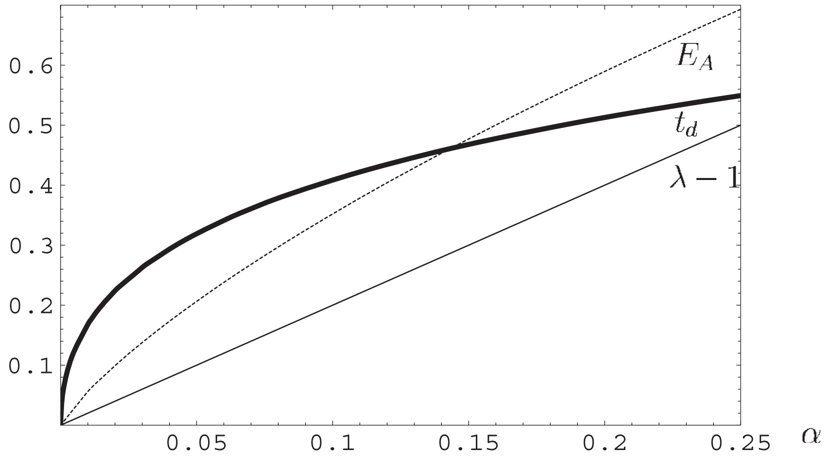

Figure 1. t_d, E_A and $\lambda - 1$ as functions of α.

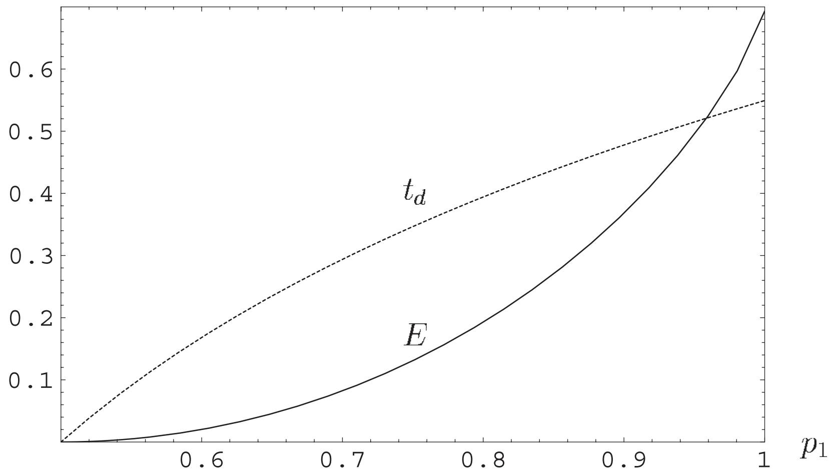

Figure 2. t_d and E as functions of $p_1 \in [1/2, 1]$.

3. Three- and four-qubit states

As we showed in the previous section, dynamical properties of a two-qubit system induced by measurement-like interaction with an environment can be used to study entanglement. In particular, the rate of decoherence classifies pure states with respect to their entanglement, and the time of disentanglement provides a practical measure of entanglement for arbitrary mixed states. Now we should like to extend these ideas and discuss entanglement of three- and four-qubit states. Since for two-qubit pure states the rate of decoherence $\lambda(P)$ and concurrence $C(P)$ are related by the equation

$$2\lambda(P) - 2 = C(P)^2$$

and so yield equivalent measures of entanglement, we propose to take the rate of stability as a measure of entanglement of pure states also in the case of three- and four-qubit systems.

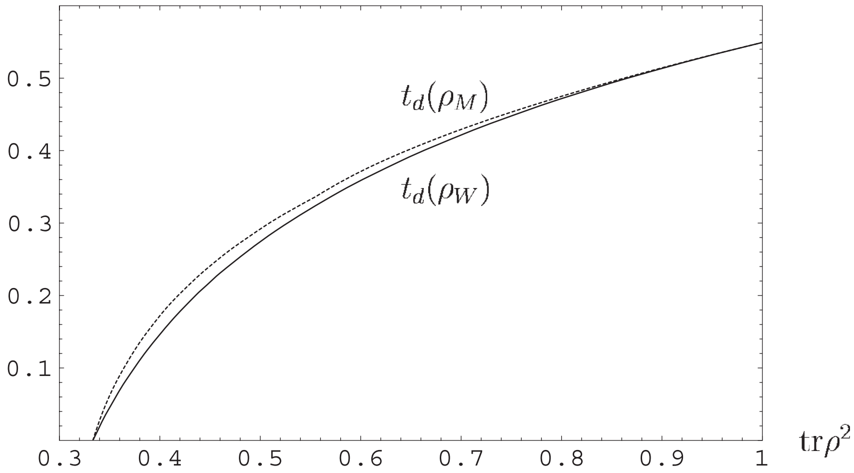

Figure 3. t_d as a function of $\mathrm{tr}\rho^2$ for ρ_M (dotted curve) and ρ_W.

3.1. Three qubits

Let $A_1,\ A_2,\ A_3$ be $\frac{1}{2}$-spin systems. The algebra of observables $\mathfrak{A}$ of the joint system $A_1A_2A_3$ is given by

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3 \qquad \mathfrak{A}_i = M_{2\times 2}$$

and is equal to the algebra of 8×8 complex matrices $M_{8\times 8}$. We consider the stability of pure states of the compound system with respect to the measurement-like interactions given by families of projectors

$$\{P_{A_1} \otimes \mathbb{I}_{A_2} \otimes I_{A_3}\}, \{\mathbb{I}_{A_1} \otimes P_{A_2} \otimes \mathbb{I}_{A_3}\}, \{\mathbb{I}_{A_1} \otimes \mathbb{I}_{A_2} \otimes P_{A_3}\}$$

where P_{A_i} are one-dimensional projectors associated with pure states of the system A_i. Generalizing the construction of the generator L given in the previous section (again we omit the coupling constant κ, which measures the strength of interaction between the system and its environment), we obtain

$$L\rho = -\mathrm{i}[H, \rho] + \sum_{k=1}^{3} \mathrm{Tr}_{A_k}\rho - 3\rho \tag{16}$$

where $\mathrm{Tr}_{A_i},\ i = 1, 2, 3$ is the conditional expectation from $M_{8\times 8}$ onto a subalgebra of $M_{8\times 8}$ isomorphic to $\mathbb{I}_{A_i} \otimes M_{4\times 4}$ given by

$$\mathrm{Tr}_{A_i}(\rho) = \tfrac{1}{2}\mathbb{I}_{A_i} \otimes \mathrm{tr}_{A_i}(\rho) \qquad i = 1, 2, 3 \tag{17}$$

and tr_{A_i} is a partial trace with respect to subsystem A_i. Now the rate of decoherence $\lambda(P)$ defined by the obvious generalization of the formula (5) is given by

$$\lambda(P) = 3 - \tfrac{1}{2} \sum_{i=1}^{3} \mathrm{tr}(\mathrm{tr}_{A_i} P)^2. \tag{18}$$

It is useful to write $\lambda(P) = \sum_{i=1}^{3} \lambda_i(P)$, where

$$\lambda_i(P) = 1 - \tfrac{1}{2}\mathrm{tr}(\mathrm{tr}_{A_i}(P))^2. \tag{19}$$

We introduce also the minimal rate of decoherence with respect to particular subsystems of $A_1A_2A_3$:

$$\lambda_{\min}(P) = \min_{1\leqslant i\leqslant 3} \lambda_i(P). \tag{20}$$

In the following we show that the rate of stability of pure states expressed in terms of λ and $\lambda_{\min}$ may serve as another measure of entanglement. More precisely, we shall say that state P is less stable than Q if $\lambda(P) \geqslant \lambda(Q)$ and $\lambda_{\min}(P) \geqslant \lambda_{\min}(Q)$.

Theorem 2. *The possible values of the two-dimensional set* $(\lambda(P),\ \lambda_{\min}(P))$ *are points of the triangle* ABC *shown in figure 4 with*

$$A = (\tfrac{3}{2}, \tfrac{1}{2}) \qquad B = (\tfrac{9}{4}, \tfrac{3}{4}) \qquad C = (2, \tfrac{1}{2}).$$

Proof. The interval AB is represented by the equation $y = 3x$. Hence

$$\lambda(P) = \sum_{i=1}^{3} \lambda_i(P) \geqslant 3\lambda_{\min}(P).$$

It is also clear that

$$\lambda_{\min}(P) \geqslant \tfrac{1}{2}$$

or, equivalently, that for all $i = 1, 2, 3$ $\mathrm{tr}(\mathrm{tr}_{A_i})^2 \leqslant 1$. Therefore, it is enough to check only that

$$\lambda_{\min}(P) \leqslant \lambda(P) - \tfrac{3}{2}$$

since the line CB is given by the equation $y = x - \frac{3}{2}$. It is equivalent to

$$\lambda(P) = \lambda_1(P) + \lambda_2(P) + \lambda_3(P) \leqslant \lambda_{\min}(P) + \tfrac{3}{2}.$$

Suppose that $\lambda_{\min}(P) = \lambda_1(P)$. Because for every $i = 1, 2, 3$

$$\mathrm{tr}(\mathrm{tr}_{A_i} P)^2 = \mathrm{tr}(\mathrm{tr}_{A_j A_k} P)^2$$

where jk are complementary to the index i, and $\mathrm{tr}_{A_j A_k} P$ is an arbitrary density matrix, so

$$\tfrac{1}{2} \leqslant \mathrm{tr}(\mathrm{tr}_{A_i} P)^2 \leqslant 1.$$

Hence

$$\tfrac{1}{2} \leqslant \lambda_i(P) \leqslant \tfrac{3}{4}$$

for any index i, and so

$$\lambda_2(P) + \lambda_3(P) \leqslant \tfrac{3}{2}.$$

□

Point A in figure 4 represents triseparable states $P = P_1 \otimes P_2 \otimes P_3$, the interval AC two-separable states, i.e. states of the form $P_i \otimes P_{jk}$. Clearly, triseparable states are the most stable ones. Points above the line AC correspond to entangled states and the point B represents the most unstable states. They are determined by the property that for all $i = 1, 2, 3,$ $\mathrm{tr}(\mathrm{tr}_{A_i} P)^2 = \frac{1}{2}$, which implies that

$$\mathrm{tr}_{A_i A_j} P = \tfrac{1}{2}\mathbb{I}$$

for any $i \neq j$. It is well known that these states coincide (up to local unitary isomorphisms) with the GHZ state $|C_3\rangle$ (GHZ class), where $|C_n\rangle = \frac{1}{\sqrt{2}}(|000\cdots0\rangle + |111\cdots1\rangle)$, and so are maximally entangled. It is also known that the GHZ class forms a five-parameter family of states [21] and we have found an explicit realization of this family

$$\Phi(\theta_1, \theta_2, \theta_3, \varphi, \psi) = \begin{pmatrix} \sin\varphi \sin\psi \mathrm{e}^{\mathrm{i}\theta_1} \\ \sin\varphi \cos\psi \mathrm{e}^{\mathrm{i}\theta_2} \\ \cos\varphi \sin\psi \mathrm{e}^{\mathrm{i}\theta_3} \\ \cos\varphi \cos\psi \mathrm{e}^{\mathrm{i}(\theta_2+\theta_3-\theta_1)} \\ \cos\varphi \cos\psi \mathrm{e}^{-\mathrm{i}(\theta_2+\theta_3-\theta_1)} \\ -\cos\varphi \sin\psi \mathrm{e}^{-\mathrm{i}\theta_3} \\ -\sin\varphi \cos\psi \mathrm{e}^{-\mathrm{i}\theta_2} \\ \sin\varphi \sin\psi \mathrm{e}^{-\mathrm{i}\theta_1} \end{pmatrix} \tag{21}$$

$$\theta_1,\ \theta_2,\ \theta_3 \in [0, 2\pi] \quad \varphi,\ \psi \in \left[0, \frac{\pi}{2}\right].$$

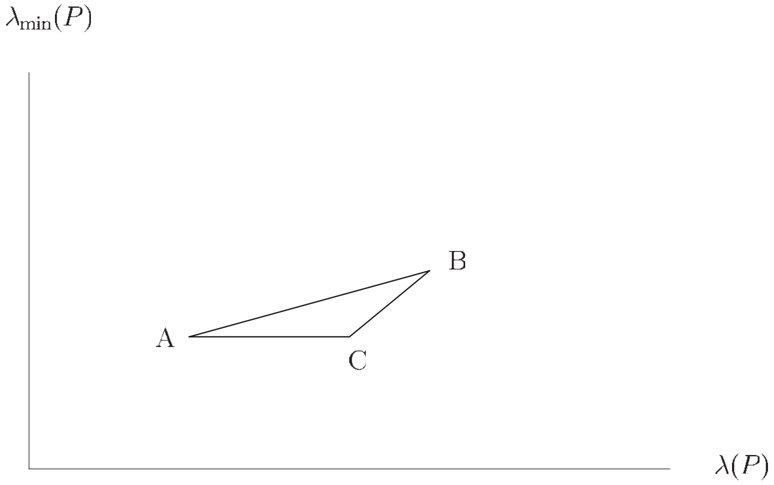

Figure 4. Range of the function $P \to (\lambda(P),\ \lambda_{\min}(P))$.

3.2. Four qubits

Now we take four copies of spin-$\frac{1}{2}$-systems, say $A_1,\ A_2,\ A_3,\ A_4$. For the states of the joint system $A_1A_2A_3A_4$ with the algebra of observables

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3 \otimes \mathfrak{A}_4$$

we consider the reduced dynamics generated by L with a dissipative part L_{D} given by measurement-like interactions with respect to families of projectors

$$\{P_{A_iA_j} \otimes \mathbb{I}_{A_k} \otimes \mathbb{I}_{A_l}\}$$

where $i < j$ and (k, l) are complementary to (i, j) in the set $\{1, 2, 3, 4\}$. A similar construction as above gives the following generator L:

$$L\rho = -\mathrm{i}[H, \rho] + \sum_{i<j} \mathrm{Tr}_{A_iA_j}\rho - 6\rho \tag{22}$$

and the rate of decoherence $\lambda(P)$ for any one-dimensional projector $P \in M_{16\times 16}$ is now given by

$$\lambda(P) = 6 - \sum_{i<j} \mathrm{tr}(\mathrm{Tr}_{A_iA_j}P)^2. \tag{23}$$

Because $\mathrm{Tr}_{A_iA_j}P = \frac{1}{4}\mathbb{I}_4 \otimes \mathrm{tr}_{A_iA_j}P$,

$$\lambda(P) = 6 - \tfrac{1}{4}\sum_{i<j} \mathrm{tr}(\mathrm{tr}_{A_iA_j}P)^2. \tag{24}$$

Since $\mathrm{tr}(\mathrm{tr}_{A_iA_j}P)^2 = \mathrm{tr}(\mathrm{tr}_{A_kA_l}P)^2$ where (kl) are complementary to (ij) in the set $\{1, 2, 3, 4\}$, formula (23) may be further simplified to

$$\lambda(P) = 6 - \tfrac{1}{2}\sum_{j=2}^{4} \mathrm{tr}(\mathrm{tr}_{A_1A_j}P)^2. \tag{25}$$

As in the three-qubit case we define the rate of decoherence with respect to a subsystem A_iA_j by

$$\lambda_{ij}(P) = 1 - \tfrac{1}{4}\mathrm{tr}(\mathrm{tr}_{A_iA_j}P)^2 \tag{26}$$

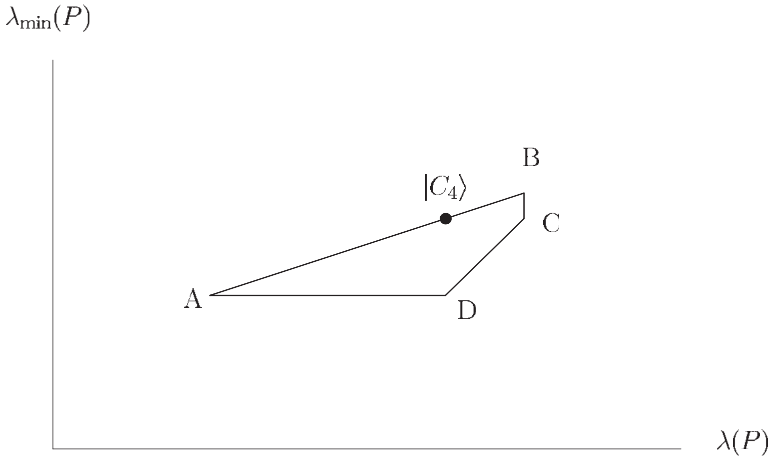

Figure 5. Range of a function $P \to (\lambda(P),\ \lambda_{\min}(P))$.

and the minimal rate with respect to any pair of subsystems

$$\lambda_{\min}(P) = \min_{i<j} \lambda_{ij} = \min_{2\leqslant j\leqslant 4} \lambda_{1j}. \tag{27}$$

Theorem 3. *The range of the function* $P \to (\lambda(P),\ \lambda_{\min}(P))$ *coincides with the set shown in figure 5 with* $A = (\frac{9}{2}, \frac{3}{4}),\ B = (\frac{11}{2}, \frac{11}{12}),\ C = (\frac{11}{2}, \frac{7}{8}),\ D = (\frac{21}{4}, \frac{3}{4})$.

Proof. See the appendix. □

Let us now describe states which correspond to the boundary points in figure 5. Point A represents four-separable states of the form $P_1 \otimes P_2 \otimes P_3 \otimes P_4$. States of the form $P_{ij} \otimes P_{kl}$ for some indices (ij) and complementary to them (kl) are represented in the interval AD. To point D corresponds states such that both P_{ij} and P_{kl} are maximally entangled as states in $\mathfrak{A}_i \otimes \mathfrak{A}_j$ ($\mathfrak{A}_k \otimes \mathfrak{A}_l$) respectively. The GHZ state $|C_4\rangle$ is represented by point $(\frac{21}{4}, \frac{7}{8})$ indicated by a dot on figure 5. Since there are states P such that both $\lambda(P) \geqslant \lambda(|C_4\rangle\langle C_4|)$ and $\lambda_{\min}(P) \geqslant \lambda_{\min}(|C_4\rangle\langle C_4|)$ so it is not the most unstable state. It is also known to be not the most entangled state. A state [22]

$$\Psi = \frac{1}{\sqrt{6}}\left(|0011\rangle + |1100\rangle + \frac{\mathrm{i}\sqrt{3}-1}{2}(|1010\rangle + |0101\rangle) - \frac{\mathrm{i}\sqrt{3}+1}{2}(|1001\rangle + |0110\rangle)\right) \tag{28}$$

is represented by point B. Taking as a measure of entanglement the entropy

$$E = E_{12} + E_{13} + E_{14}$$

where

$$E_{ij} = -\operatorname{tr}\,(\operatorname{tr}_{A_iA_j} P\ \log_2 \operatorname{tr}_{A_iA_j} P)$$

we obtain that

$$E(|\Psi\rangle\langle\Psi|) = 3 + \tfrac{3}{2}\log_2 3.$$

It was argued in [22] that this is the maximal possible value of the entropy E. Hence, since all λ_{ij} are invariant with respect to the local unitary transformations, we may conclude that maximally entangled states are also maximally unstable. It is worth noting that the above diagrams are specific to the type of evolution considered here. In general, for the dynamics given by a quantum dynamical (Markov) semi-group, there are present so-called decoherence free subspaces also referred to as noiseless quantum codes (see for example [23] and references therein). In such a case there exist states with the decoherence rate equal to zero and so with an infinite time of disentanglement. However, since even a unitary dynamics, for which all states are immune to decoherence, may transform entangled to separable (and vice versa) states, such a general framework of quantum dynamical semi-groups cannot be effectively used to measure the degree of entanglement of all states.

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Appendix

Proof of theorem 3. We have to prove five bounds on possible values of $(\lambda(P), \lambda_{\min}(P))$. First we show that $\lambda(P) \geqslant \frac{9}{2}$. This is clear because for any $j \in \{2, 3, 4\}$, $\mathrm{tr}(\mathrm{tr}_{A_1 A_j} P)^2 \leqslant 1$. The lower bound on $\lambda_{\min}(P)$, that is $\lambda_{\min}(P) \geqslant \frac{3}{4}$, follows from the same argument. Because $\lambda(P) = \sum_{i<j} \lambda_{ij}(P)$ and the interval AB is given by equation $y = \frac{1}{6}x$, the obvious inequality $\lambda(P) \geqslant 6\,\lambda_{\min}(P)$ gives the appropriate upper bound on $\lambda_{\min}(P)$. The interval DC lies on the line $y = \frac{1}{2}x - \frac{15}{8}$, hence we must show that

$$\lambda_{\min}(P) \geqslant \tfrac{1}{2}\lambda(P) - \tfrac{15}{8}. \tag{29}$$

Suppose that $\lambda_{\min}(P) = \lambda_{12}(P)$. Then we may rewrite (28) as

$$\lambda_{12}(P) \geqslant \lambda_{12}(P) + \lambda_{13}(P) + \lambda_{14}(P) - \tfrac{15}{8} \tag{30}$$

or

$$\lambda_{13}(P) + \lambda_{14}(P) \leqslant \tfrac{15}{8}. \tag{31}$$

Because for any $i < j$, $\mathrm{tr}(\mathrm{tr}_{A_i A_j} P)^2 \geqslant \frac{1}{4}$, so $\lambda_{ij}(P) \leqslant \frac{15}{16}$ and thus inequality (30) follows. Finally, we show that

$$\lambda(P) \leqslant \tfrac{11}{2}. \tag{32}$$

Let $\rho = \mathrm{tr}_{A_1} P$. This is an 8×8 density matrix. Then inequality (31) is equivalent to the following one:

$$\sum_{j=2}^{4} \mathrm{tr}(\mathrm{tr}_{A_j} \rho)^2 \geqslant 1. \tag{33}$$

Let us first show that there are two one-dimensional projectors $P, Q \in M_{8\times 8}(\mathbb{C})$ such that $\rho = \frac{1}{2}(P + Q)$. Clearly, $\rho = \alpha \tilde{P} + (1-\alpha)\tilde{Q}$ for some $\tilde{P}$, $\tilde{Q}$, $\alpha \in [0, 1]$ and there exists $U \in SU(8)$ such that $\rho = U\rho_0 U^*$, where

$$\rho_0 = \mathrm{diag}\ (a, 1-a, 0, \ldots, 0).$$

Suppose that

$$\psi = \left(\sqrt{a}, \sqrt{1-a}, 0, \ldots, 0\right) \in \mathbb{C}^8$$

$$\phi = \left(\sqrt{a}, -\sqrt{1-a}, 0, \ldots, a\right) \in \mathbb{C}^8$$

and

$$P_\psi = |\psi\rangle\langle\psi| \qquad P_\phi = |\phi\rangle\langle\phi|.$$

Then $\rho_0 = \frac{1}{2}P_\psi + \frac{1}{2}P_\phi$. Hence

$$\rho = U(\tfrac{1}{2}P_\psi + \tfrac{1}{2}P_\phi)U^* = \tfrac{1}{2}P + \tfrac{1}{2}Q$$

where $P = UP_\psi U^*$ and $Q = UP_\phi U^*$. Therefore, inequality (32) is equivalent to

$$\frac{1}{2}\sum_{j=2}^{4}\mathrm{tr}(\mathrm{tr}_{A_j}P)^2 + \frac{1}{2}\sum_{j=2}^{4}\mathrm{tr}(\mathrm{tr}_{A_j}Q)^2 + \sum_{j=2}^{4}\mathrm{tr}[(\mathrm{tr}_{A_j}P)(\mathrm{tr}\,A_jQ)] \geqslant 2. \qquad (34)$$

To simplify notation we define

$$h(P,Q) = \sum_{j=2}^{4}\mathrm{tr}[(\mathrm{tr}_{A_j}P)(\mathrm{tr}_{A_j}Q)]$$

and

$$k(P,Q) = \sum_{j=2,j<k}^{4}\mathrm{tr}[(\mathrm{tr}_{A_jA_k}P)(\mathrm{tr}_{A_jA_k}Q)].$$

Clearly

$$h(P,P) = \sum_{j=2}^{4}\mathrm{tr}(\mathrm{tr}_{A_j}P)^2 = \sum_{j=2,j<k}\mathrm{tr}(\mathrm{tr}_{A_jA_k}P)^2 = k(P,P).$$

Suppose that

$$h(P,Q) \geqslant k(P,Q) - 1.$$

Then

$$\begin{aligned}
\tfrac{1}{2}(h(P,P)+h(Q,Q)) + h(P,Q) &\geqslant \tfrac{1}{2}\Bigg[\sum_{j<k}\mathrm{tr}(\mathrm{tr}_{A_jA_k}P)^2 + \sum_{j<k}\mathrm{tr}(\mathrm{tr}_{A_jA_k}Q)^2 \\
&\quad +2\sum_{j<k}\mathrm{tr}(\mathrm{tr}_{A_jA_k}P)(\mathrm{tr}_{A_jA_k}Q) - 2\Bigg] \\
&= \tfrac{1}{2}\sum_{j<k}\mathrm{tr}(\mathrm{tr}_{A_jA_k}(P+Q))^2 - 1 \\
&= 2\sum_{j<k}\mathrm{tr}\left(\mathrm{tr}_{A_jA_k}\left(\frac{P+Q}{2}\right)\right)^2 - 1 \geqslant 2
\end{aligned}$$

since

$$\mathrm{tr}\left(\mathrm{tr}_{A_jA_k}\left(\frac{P+Q}{2}\right)^2\right) \geqslant \frac{1}{2}$$

for any (j,k), and so inequality (31) follows. Therefore, to complete the proof it suffices to show the following lemma. □

Lemma. *For any pair of one-dimensional projectors $P, Q \in M_{8\times 8}(\mathbb{C})$ we have that*

$$h(P,Q) \geqslant k(P,Q) - 1.$$

Proof. Let

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{18} \\ \cdots & \cdots & \cdots & \cdots \\ p_{81} & p_{82} & \cdots & p_{88}\end{pmatrix} \qquad Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{18} \\ \cdots & \cdots & \cdots & \cdots \\ q_{81} & q_{82} & \cdots & q_{88}\end{pmatrix}$$

where $p_{ij} = a_i\overline{a_j},\ q_{ij} = b_i\overline{b_j}$ and $\sum_{i=1}^{8} |a_i|^2 = \sum_{i=1}^{8} |b_i|^2 = 1$. Because

$$\mathrm{tr}_{A_1} P = \begin{pmatrix} p_{11}+p_{55} & p_{12}+p_{56} & p_{13}+p_{57} & p_{14}+p_{58} \\ p_{21}+p_{65} & p_{22}+p_{66} & p_{23}+p_{67} & p_{24}+p_{68} \\ p_{31}+p_{75} & p_{32}+p_{76} & p_{33}+p_{77} & p_{34}+p_{78} \\ p_{41}+p_{85} & p_{42}+p_{86} & p_{43}+p_{87} & p_{44}+p_{88} \end{pmatrix}$$

$$\mathrm{tr}_{A_2} P = \begin{pmatrix} p_{11}+p_{33} & p_{12}+p_{34} & p_{15}+p_{37} & p_{16}+p_{38} \\ p_{21}+p_{43} & p_{22}+p_{44} & p_{25}+p_{47} & p_{26}+p_{48} \\ p_{51}+p_{73} & p_{52}+p_{74} & p_{55}+p_{77} & p_{56}+p_{78} \\ p_{61}+p_{83} & p_{62}+p_{84} & p_{65}+p_{87} & p_{66}+p_{88} \end{pmatrix}$$

$$\mathrm{tr}_{A_3} P = \begin{pmatrix} p_{11}+p_{22} & p_{13}+p_{24} & p_{15}+p_{26} & p_{17}+p_{28} \\ p_{31}+p_{42} & p_{33}+p_{44} & p_{35}+p_{46} & p_{37}+p_{48} \\ p_{51}+p_{62} & p_{53}+p_{64} & p_{55}+p_{66} & p_{57}+p_{68} \\ p_{71}+p_{82} & p_{73}+p_{84} & p_{75}+p_{86} & p_{77}+p_{88} \end{pmatrix}$$

and

$$\mathrm{tr}_{A_1A_2} P = \begin{pmatrix} p_{11}+p_{33}+p_{55}+p_{77} & p_{12}+p_{34}+p_{56}+p_{78} \\ p_{21}+p_{43}+p_{65}+p_{87} & p_{22}+p_{44}+p_{66}+p_{88} \end{pmatrix}$$

$$\mathrm{tr}_{A_1A_3} P = \begin{pmatrix} p_{11}+p_{22}+p_{55}+p_{66} & p_{13}+p_{24}+p_{57}+p_{68} \\ p_{31}+p_{42}+p_{75}+p_{86} & p_{33}+p_{44}+p_{77}+p_{88} \end{pmatrix}$$

$$\mathrm{tr}_{A_2A_3} P = \begin{pmatrix} p_{11}+p_{22}+p_{33}+p_{44} & p_{15}+p_{26}+p_{37}+p_{48} \\ p_{51}+p_{62}+p_{73}+p_{84} & p_{55}+p_{66}+p_{77}+p_{88} \end{pmatrix}$$

by direct calculation we obtain that

$$k(P,Q) - h(P,Q) = 1 - |\langle\vec{a},\ \vec{b}\rangle|^2 - |\langle\vec{a},\ R\vec{b}\rangle|^2 \tag{35}$$

where $\vec{a} = (a_1, \ldots, a_8),\ \vec{b} = (b_1, \ldots, b_8)$ and R is an antiunitary operator on $\mathbb{C}^8$ defined as follows:

$$R(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (\overline{a}_8, -\overline{a}_7, -\overline{a}_6, \overline{a}_5, -\overline{a}_4, \overline{a}_3, \overline{a}_2, -\overline{a}_1).$$

Because vectors $\vec{b}$ and $R\vec{b}$ are orthogonal and normalized,

$$0 \leqslant k(P,Q) - h(P,Q) \leqslant 1. \tag{36}$$

□

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